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WAVE PROPAGATION IN TRANSVERSELY ISOTROPIC, LAYERED CYLINDERS

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ABSTRACT

The propagation of infinite trains of symmetric harmonic waves traveling in infinitely long, right circular cylindrical shells is investigated on the basis of the three-dimensional theory of elasticity. The shells are assumed made of three concentric, transversely isotropic cylinders, each of different materials, bonded perfectly at their interfaces. The frequency equation is established by representing the displacement field in each cylinder in terms of potential functions and satisfying the Navier equations of motion and the boundary and interface conditions of the cylinder.

The frequency equation has been programmed for numerical evaluation on an IBM 7044/7094 DCS computer, and the influence of the mechanical properties of the layers on the frequencies of the first few modes is investigated.

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LIST OF SYMBOLS

$z,\ r,\ \theta$	Cylindrical coordinate
$\bar{r}$	Non-dimensionalyzed radial coordinate
a, b, c, d	Radii of the shell layers (see fig. 1)
$\bar{a}, \bar{b}, \bar{c}, \bar{d}$	Nondimentionalyzed Radii of the shell layers
$h^{(i)}$	Thickness of i^{th} layer
$u_r,\ u_z$	displacement of components in the r and z respectively
e_i	Component of strain, defined in eq. [1].
τ	Component of stree defined in Eq. [1].
$\rho^{(i)}$	density
c_{ij}	Elastic constants
d_j	Nondimensional elastic constants
k	Wavenumber in the axial direction
$\zeta = \frac{kh^{(2)}}{\pi}$	non-dimensionalized frequency
ω	circular frequency
$\Omega = \frac{\omega h^{(2)}}{\pi}$	non-dimensionalized wavenumber
t	Time
$v_s^{(i)} = \frac{c_{44}^{(i)}}{\rho^{(i)}}$ $v_d^{(i)} = \frac{c_{33}^{(i)}}{\rho^{(i)}}$	"velocities"
$a^{(i)},\ b^{(i)}$	nondimensional velocity ratios
$p,\ q$	radial wave numbers (see equation [8])
$\alpha,\ \beta$	see equations [19] and [20]
$m = \rho \frac{\omega^2}{k^2}$	
$Z_o,\ W_o$	zero order Bessel function of the first or second kind respectively

INTRODUCTION

The frequency equation for harmonic waves traveling in traction-free infinitely long, isotropic circular cylindrical shells has been established on the basis of the three-dimensional theory of elasticity and has been evaluated numerically by Gazis (4)[1] and Greenspon (5). More recently, Mirsky investigated the propagation of harmonic waves in circular cylindrical shells made of transversely isotropic and of orthotropic matericals (9) (10).

The increasing demand for structural components of aerospace vehicles having a high strength to weight ratio and being capable of withstanding high temperatures, has resulted in extensive use of multi-layered shells and in considerable interest in the propagation of harmonic waves in such shells. Armenàkas (1) (2) presented a unified treatment, on the basis of the theory of elasticity, for harmonic waves of an arbitrary number of cicumferential nodes traveling in two layered isotropic shells. Keck and Armenàkas (7) investigated the propagation of harmonic axisymmetric waves in sandwich isotropic shells. Moreover, a number of approximate theories for two and three layered shells were established (3) (6) (11).

In this investigation, the frequency equation for propagation of trains of axisymmetric nontorsional harmonic waves in infinitely long shells, made of three concentric cylinders of different transversely

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[1] Numerals in parentheses refer to References at the end of the report

isotropic materials, is derived on the basis of the linear theory of elasticity. It is shown that as in the case of isotropic shells, for waves having infinite axial wave length, the frequency equation degenerates into two independent equations for uncoupled longitudinal shear and uncoupled radial motion.

The frequency equation has been programmed for numerical evaluation on an IBM 7044/7094 DCS computer, and the influence of the mechanical properties on the frequencies of the first few modes is investigated.

SOLUTIONS OF THE EQUATIONS OF MOTION

In the ensuing derivations, the following notation for the components of stress and stain will be used

$$\tau_{11}=\tau_1,\ \tau_{22}=\tau_2;\ \tau_{33}=\tau_3,\ \tau_{23}=\tau_4,\ \tau_{31}=\tau_5,\ \tau_{12}=\tau_6, \qquad [1]$$
$$e_{11}=e_1,\ e_{22}=e_2,\ e_{33}=e_3,\ 2e_{23}=e_4,\ 2e_{31}=e_5,\ 2e_{12}=e_6\ .$$

In terms of this notation, Hooke's law for a general anistropic body may be written as

$$\tau_i = c_{ij}e_j \quad (i,\ j = 1,\ 2,\ \ldots,\ 6). \qquad [2]$$

For a transversely isotropic body, in particular, these relations reduce to

$$\begin{aligned} \tau_1 &= c_{11}e_1 + c_{12}e_2 + c_{13}e_3, \\ \tau_2 &= c_{12}e_1 + c_{11}e_2 + c_{13}e_3, \\ \tau_3 &= c_{13}e_1 + c_{13}e_2 + c_{33}e_3, \\ \tau_4 &= c_{44}e_4,\quad \tau_5 = c_{44}e_5,\quad \tau_6 = c_{66}e_6\ , \end{aligned} \qquad [3]$$

where

$$c_{66} = \tfrac{1}{2}(c_{11} - c_{12}).$$

The assumption that the strain energy density is a positive definite quadratic function of the components of strain imposes the following restrictions on the elastic constants

$$\begin{aligned} &c_{11} > 0,\ c_{33} > 0,\ c_{44} > 0, \\ &c_{11}^2 - c_{12}^2 > 0,\ c_{11}\,c_{33} - c_{13}^2 > 0, \\ &c_{11}\,c_{33} + c_{12}\,c_{33} - 2c_{13}^2 > 0. \end{aligned} \qquad [4]$$

Equations [3] reduce to those for an isotropic body by employing the following relations between the elastic constants

$$c_{33} = c_{11}, \quad c_{12} = c_{13}, \quad c_{44} = c_{66}.$$

The components of stress may be obtained in terms of the components of displacement by substituting the strain-displacement relations into the constituative equations [3]. These, in turn, may be substituted into the stress equations of motion to obtain the following displacement equations of motion

$$c_{11}(u_{r,rr} + u_{r,r}/r - u_r/r^2) + c_{66}\, u_{r,\theta\theta}/r^2 + c_{44}u_{r,zz} +$$
$$(c_{66} + c_{12})u_{\theta,r\theta}/r - (c_{66} + c_{11})u_{\theta,\theta}/r^2 + (c_{13} + c_{44})u_{z,rz} = \rho\ddot{u}_r,$$

$$(c_{12} + c_{66})u_{r,r\theta}/r + (c_{11} + c_{66})u_{r,\theta}/r^2 + c_{66}(u_{\theta,rr} + u_{\theta,r}/r - u_\theta/r^2)$$
$$+ c_{11}\, u_{\theta,\theta\theta}/r^2 + c_{44}u_{\theta,zz} + (c_{13} + c_{44})u_{z,\theta z}/r = \rho\ddot{u}_\theta, \qquad [5]$$

$$(c_{13} + c_{44})\ (u_{r,rz} + u_{r,z}/r + u_{\theta,rz}/r) + c_{13}u_{z,zz} +$$
$$c_{44}(u_{z,rr} + u_{z,r}/r + u_{z,\theta\theta}/r^2) = \rho\ddot{u}_z.$$

Here ρ is mass density per unit volume; subscripts preceeded by a comma denote differentiation with respect to the space coordinates. The dot indicates differentiation with respect to time. It can be shown (8) that for axisymmetric motion, Eqs. [5] are satisfied by a displacement field of the following form

$$u_r = (c_{13} + c_{44})\phi(r)_{,r} \cos(kz - \omega t), \qquad [6]$$

$$u_z = \frac{-1}{k}\left[c_{11}\nabla_1^2\,\phi(r) + k^2\left(\frac{\rho\omega^2}{k^2} - c_{44}\right)\phi(r)\right]\sin(kz - \omega t), \qquad [6\text{ cont}]$$

where the potential function $\phi(r)$ must satisfy the equation

$$(\nabla_1^2 + p^2)\,(\nabla_1^2 + q^2)\,\phi(r) = 0. \qquad [7]$$

Here

$$\nabla_1^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr},$$

and ω and k are the frequency and the axial wave number of the wave, respectively. The radial wave numbers p and q are given by

$$\left.\begin{matrix} p^2 \\ q^2 \end{matrix}\right\} = \frac{k^2}{2c_{11}c_{44}}\left[A \pm \sqrt{A^2 - B}\,\right], \qquad [8]$$

where

$$\begin{aligned} A &= c_{11}(m - c_{33}) + c_{44}(m - c_{44}) + (c_{13} + c_{44})^2, \\ B &= 4c_{11}\,c_{44}(m - c_{33})\,(m - c_{44}), \end{aligned} \qquad [9]$$

and

$$m = \rho\omega^2/k^2.$$

The plus sign in equation [7] refers to p^2. Notice, when the radicand $A^2 - B$ is negative, that the radial wave numbers p and q become complex. Thus for a certain range of the values of the elastic constants there is a range of values of real m for which the radial wave numbers p and q become complex. In the case of isotropic elastic shells, the radial wave numbers do not assume complex values for any real value of m. Thus, it seems appropriate to classify transversely isotropic materials as (a) <u>less anisotropic</u> if their mechanical properties are such that p and q do not become complex for any value of m, (b) <u>more anisotropic</u> if their mechanical properties are such that p and q may become complex for a certain range of m.

The operators in equation [7] are Bessel operators and, consequently, their solution is given in terms of the zero order Bessel functions of the first and second kind with real, complex or imaginary arguments, depending on whether p and q are real, complex or imaginary. Therefore in order to specify the real solutions for ϕ, and the displacement and stress fields, it is necessary to establish the range of the material properties and the range of values of the wave parameters (positive values of ω and k^2) for which p, q are real, imaginary or complex.

The radicand $A^2 - B$ of equation [8] vanishes if m satisfies the equation

$$\left.\begin{matrix} m_2 \\ m_1 \end{matrix}\right\} = \frac{(c_{11}c_{33} - c_{44}^2)(c_{11} - c_{44}) - (c_{11} + c_{44})(c_{13} + c_{44})^2}{(c_{11} - c_{44})^2} \pm$$

$$\pm \frac{2(c_{13} + c_{44})}{(c_{11} - c_{44})^2} \sqrt{c_{11}\, c_{44} \left[(c_{13} + c_{44})^2 - (c_{11} - c_{44})(c_{33} - c_{44}) \right]} \quad . \qquad [10]$$

For ordinary engineering materials, it may be assumed that $c_{33} > c_{44}$. On this basis it can be shown that the radicand $A^2 - B$ cannot vanish for values of m greater than c_{44}. Thus, for a given material, the radicand $A^2 - B$ becomes negative for values of m satisfying

$$m_1 < m < m_2 \leq c_{44} \qquad \text{if } m_1 > 0,$$

$$0 \leq m < m_2 \leq c_{44} \qquad \text{if } m_1 < 0.$$

If the inequality, $A^2 - B \leq 0$, is solved for c_{11}, the following inequality results

$$\frac{[(c_{13} + c_{44}) - \sqrt{c_{44}(c_{44} - m)}]^2}{(c_{33} - m)} \leq c_{11} \leq \frac{[(c_{13} + c_{44}) + \sqrt{c_{44}(c_{44} - m)}]^2}{(c_{33} - m)} \quad . \qquad [12]$$

The cross-hatched region in Fig. 1 represents the locus of the values of m and c_{11} satisfying this inequality. Note, that when $m = 0$, inequality [12] reduces to

$$\frac{c_{13}^2}{c_{33}} \le c_{11} \le \frac{(c_{13} + 2c_{44})^2}{c_{33}}, \qquad [13]$$

whereas, when $m = c_{44}$, inequality [12] yields

$$c_{11} = \frac{(c_{13} + c_{44})^2}{c_{33} - c_{44}} . \qquad [14]$$

The value of m corresponding to the maximum value of c_{11}, for which the radicant $A^2 - B$ vanishes, can be obtained by setting the derivative of τ_{11} with respect to m (see fig. 1) equal to zero.

$$\frac{d(\tau_{11})}{dm} = \frac{d}{dm}\left(\frac{[c_{13} + c_{44} + [c_{44}(c_{44} - m)]^{\frac{1}{2}}]^2}{c_{33} - m}\right) = 0. \qquad [15]$$

This results in

$$m\Big|_{\text{for } \tau_{11} \max} = c_{44}\left[1 - \frac{(c_{33} - c_{44})^2}{(c_{13} + c_{44})^2}\right],$$

and [16]

$$(\tau_{11})_{max} = c_{44} + \frac{(c_{13} + c_{44})^2}{c_{33} - c_{44}} .$$

From equation [10] it can be seen that the maximum value of τ_{11} is also obtained when $m = m_1 = m_2$. From the aforegoing discussion we may conclude that if the elastic constants of a material satisfy the relation

$$c_{11} \ge c_{44} + \frac{(c_{13} + c_{44})^2}{c_{33} - c_{44}}, \qquad [14]$$

the radicant $A^2 - B$ is positive for all values of m and, consequently, the radial wave numbers p and q do not become complex for any values of m. In this case, the material has been classified as less anisotropic. If, however, the elastic constants of a material do not satisfy inequality [17], then for a certain range of values of m the radicant $A^2 - B$ will be negative and p and q will be complex. In this case, the material has been classified as more anisotropic. Referring to equation [8], it can be seen that p^2 is the conjugate of q^2. Consequently, two pairs of complex values of p and q are found. It can be shown that any two of the four roots satisfy the requirements of the equations of motion. The value of q used in this analysis is the negative complex conjugate of p.

From Eqs. [9], it can be deduced that for $m > c_{33}$ the radial wave numbers p and q are real, while for $c_{33} > m > c_{44}$, p is real and q is imaginary. Finally, for $c_{44} > m$, the non-complex values of p and q are both real if $A > 0$ or both imaginary if $A < 0$. The parameter A vanishes if

$$m_3 = \frac{c_{11} c_{33} + c_{44}^2 - (c_{13} + c_{44})^2}{c_{11} + c_{44}} . \qquad [18]$$

This relation is plotted in Fig. 1. For values of m less than m_3, A is negative.

It can be seen, that for less anisotropic materials and for more anisotropic materials with elastic constants satisfying the relation

$$c_{11} > \frac{(c_{13} + c_{44})^2}{c_{33} - c_{44}}$$

the parameter A is negative for all values of m smaller than c_{44}. For these materials, for $c_{44} > m$, the non-complex values of p and q are imaginary. For more anisotropic materials with elastic constants satisfying the relation $c_{11} < (c_{13} + c_{44})^2/(c_{33} - c_{44})$, the parameter A is positive for $c_{44} > m > m_2$. Therefore, in this range, p and q are both real. On the basis of the foregoing discussion, the radial wave numbers p and q may be written in the following form

$$p = (\varepsilon_1)^{\frac{1}{2}} k\alpha \qquad q = (\varepsilon_2)^{\frac{1}{2}} k\beta. \qquad [19]$$

For less anisotropic materials and for more anisotropic materials, if $m < m_1$ or $m > m_2$, the parameters α and β are given by

$$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\} = \frac{1}{\sqrt{2c_{11}\, c_{44}}} \left| A \pm \sqrt{A^2 - B} \right|^{\frac{1}{2}} , \qquad [20]$$

and alternatively, for more anisotropic materials with $m_1 < m < m_2$ (here m_1 is replaced by 0 if $m_1 < 0$) α and β are taken as

$$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\} = \tilde{B}^{\frac{1}{4}} (\pm \cos\theta + i \sin\theta) , \qquad [21]$$

with

$$\tilde{B} = \left| \frac{1}{c_{11}} (m - c_{33}) \frac{1}{c_{44}} (m - c_{44}) \right| , \qquad [22]$$

and

$$\theta = \frac{1}{2} \tan^{-1}\left\{ \left| \frac{\sqrt{B - A^2}}{A} \right| \right\}.$$

For materials with $c_{11} > (c_{13} + c_{44})^2/(c_{33} - c_{44})$, the sign factors ε_i (i=1,2) are assigned the values

$$\varepsilon_1 = \begin{cases} 1 & \text{if} \quad m > c_{44} \\ -1 & \text{if} \quad c_{44} > m \geq m_2 \\ 1 & \text{if} \quad m_2 > m > m_1 \\ -1 & \text{if} \quad m_1 \geq m \geq 0, \end{cases} \qquad [23]$$

$$\varepsilon_2 = \begin{cases} 1 & \text{if} \quad m > c_{33} \\ -1 & \text{if} \quad c_{33} > m \geq m_2 \\ 1 & \text{if} \quad m_2 > m > m_1 \\ -1 & \text{if} \quad m_1 \geq m \geq 0, \end{cases} \qquad [23]$$

whereas, for materials with $c_{11} < (c_{13} + c_{44})^2 / (c_{33} - c_{44})$, the factors ε_i $(i = 1, 2)$ assume the values

$$\varepsilon_1 = \begin{cases} 1 & \text{if} \quad m > m_1 \\ -1 & \text{if} \quad m_1 \geq m > 0, \end{cases}$$

and [24]

$$\varepsilon_2 = \begin{cases} 1 & \text{if} \quad m > c_{33} \\ -1 & \text{if} \quad c_{33} > m > c_{44} \\ 1 & \text{if} \quad c_{44} > m > m_1 \\ -1 & \text{if} \quad m_1 \geq m \geq 0. \end{cases}$$

The solution for ϕ can now be written as

$$\phi(r) = \frac{k}{c_{13} + c_{44}} \left[A_1 Z_o (k\alpha r) + B_1 W_o (k\alpha r) \right] + \qquad [25]$$

$$+ A_2 Z_o (k\beta r) + B_2 W_o (k\beta r) .$$

Here, Z_o and W_o are regular or modified Bessel functions of the first and second kind, respectively, depending on whether ε_i is 1 or -1.

DERIVATION OF THE FREQUENCY EQUATION

Consider a shell made of three concentric cylinders of different materials perfectly bonded at the interfaces. The solution obtained in the previous section may be applied to each layer of the shell. The material properties of the layers will be identified with the superscripts "(1)", "(2)", and "(3)" for the inner, middle and outer layers, respectively (see Fig. 2).

For convenience, the following non-dimensionalized parameters are introduced.

$$\bar{r}, \bar{a}, \bar{b}, \bar{c}, \bar{d} = (r, a, b, c, d)\frac{\pi}{h^{(2)}}, \quad \zeta = kh^{(2)}/\pi ,$$

$$\Omega = \omega h^{(2)}\sqrt{\rho^{(2)}}/\pi\sqrt{c_{44}^{(2)}}, \quad \bar{u}_j = u_j\frac{\pi}{h^{(2)}} \; (j = r, z),$$

$$d_1^{(i)} = \frac{c_{11}^{(i)}}{c_{44}^{(2)}}, \quad d_2^{(i)} = \frac{c_{12}^{(i)}}{c_{44}^{(2)}}, \quad d_3^{(i)} = \frac{c_{33}^{(i)}}{c_{44}^{(2)}}$$

$$d_4^{(i)} = \frac{c_{44}^{(i)}}{c_{44}^{(2)}}, \quad d_5^{(i)} = \frac{c_{13}^{(i)}}{c_{44}^{(2)}}, \qquad (i = 1,2,3) \qquad [26]$$

$$d_6 = \tfrac{1}{2}\frac{c_{11}^{(i)} - c_{12}^{(i)}}{c_{44}^{(2)}}, \quad \bar{\rho}^{(i)} = \frac{\rho^{(i)}}{\rho^{(2)}}$$

The ratios $[c_{44}^{(i)}/\rho^{(i)}]^{\frac{1}{2}}$ and $[c_{33}^{(i)}/\rho^{(i)}]^{\frac{1}{2}}$ are referred to as "velocities" and are denoted by $v_s^{(i)}$ and $v_d^{(i)}$, respectively. In non-dimensionalized form, these velocities will be taken as

$$a^{(i)} = \frac{v_s^{(i)}}{v_s^{(2)}} \quad \text{and} \quad b^{(i)} = \frac{v_d^{(i)}}{v_s^{(2)}}, \qquad (i = 1,2,3). \quad [27]$$

The radial wave numbers may now be re-written in terms of the non-dimensionalized parameters as

$$\bar{p}^{(i)} = \sqrt{\varepsilon_1^{(i)}} \quad \zeta\alpha^{(i)} ,$$

$$\bar{q}^{(i)} = \sqrt{\varepsilon_2^{(i)}} \quad \zeta\beta^{(i)} , \qquad [28]$$

where, for less anisotropic materials and for more anisotropic

materials if $m < m_1$ or $m > m_2$, the parameters $\underline{\alpha}^{(i)}$ and $\underline{\beta}^{(i)}$ are given by

$$\left.\begin{array}{l}\underline{\alpha}^{(i)}\\ \underline{\beta}^{(i)}\end{array}\right\} = \frac{1}{\sqrt{2}}\left|\underline{A}^{(i)} \pm \sqrt{\underline{A}^{(i)^2} - \underline{B}^{(i)}}\right|^{\frac{1}{2}}, \qquad [29]$$

with

$$\underline{A}^{(i)} = \frac{d_3^{(i)}}{d_4^{(i)}}\left(\frac{\Omega^2}{b^{(i)^2}\zeta^2} - 1\right) + \frac{d_4^{(i)}}{d_1^{(i)}}\left(\frac{\Omega^2}{a^{(i)^2}\zeta^2} - 1\right) + \frac{d_4^{(i)}}{d_1^{(i)}}\left(\frac{d_5^{(i)}}{d_4^{(i)}} + 1\right)^2,$$

$$\underline{B}^{(i)} = 4\,\frac{d_3^{(i)}}{d_1^{(i)}}\left(\frac{2}{a^{(i)^2}\zeta^2} - 1\right)\left(\frac{2}{b^{(i)^2}\zeta^2} - 1\right). \qquad [30]$$

For "more anisotropic" materials with $m_1 < m < m_2$ (m_1 replaced by 0 if $m_1 < 0$), the parameters $\underline{\alpha}^{(i)}$ and $\underline{\beta}^{(i)}$ are given by

$$\left.\begin{array}{l}\underline{\alpha}^{(i)}\\ \underline{\beta}^{(i)}\end{array}\right\} = \underline{\tilde{B}}^{(i)^{\frac{1}{4}}}\left(\pm \cos\theta^{(i)} + \sin\underline{\theta}^{(i)}\right), \qquad [31]$$

with

$$\underline{\tilde{B}}^{(i)} = \left|\tfrac{1}{4}\,\underline{B}^{(i)}\right|,$$

$$\underline{\theta}^{(i)} = \tfrac{1}{2}\tan^{-1}\left|\frac{\left|\underline{B}^{(i)} - \underline{A}^{(i)^2}\right|^{\frac{1}{2}}}{\underline{A}^{(i)}}\right|. \qquad [32]$$

The solution of the equations of motion given by equations [6] and [25] may be applied directly to each layer of the cylinder. Thus, the complete solution for the three-layered cylinder will contain twelve integration constants $A_j^{(i)}$, $B_j^{(i)}$, ($i=1,2,3$; $j=1,2$). These constants may be evaluated by requiring the solution to satisfy the following boundary and interface conditions of the cylinder.

$$\tau_{rj}^{(i)} = 0 \text{ at } \bar{r} = \bar{a}, \quad \tau_{rj}^{(3)} = 0 \text{ at } \bar{r} = \bar{d} \qquad [33]$$

$$(j=r,z)$$

$$u_j^{(i)} = u_j^{(i+1)} \qquad \text{at } \bar{r} = \bar{b} \text{ if } i = 1,$$

[33 cont]

$$\tau_{rj}^{(i)} = \tau_{rj}^{(i+1)} \qquad \text{or } \bar{r} = \bar{c} \text{ if } i = 2,$$

$$(j = r, z).$$

By substitution of the components of displacement and stress into equations [33], a set of twelve homogeneous, linear, algebraic equations are obtained. For a non-trivial solution, the determinant of the coefficients of the $A_j^{(i)}$ and $B_j^{(i)}$ must vanish resulting in the following frequency equation

$$|C_{ij}| = 0 \qquad (i,j = 1,2, \ldots, 12). \qquad [34]$$

The non-zero elements of the determinant are given as

$$C_{11} = 2\, d_6^{(1)}\, \varepsilon_1^{(1)} \zeta^2\, \underline{\alpha}^{(1)}\, \bar{a}\, Z_1 (\zeta\, \underline{\alpha}^{(1)} \bar{a}) -$$

$$- \frac{\zeta^3 d_4^{(1)} \bar{a}^2}{d_5^{(1)} + d_4^{(1)}} \left(d_1^{(1)} \varepsilon_1^{(1)}\, \underline{\alpha}^{(1)2} + d_5^{(1)} \frac{\Omega^2}{a^{(1)2} \zeta^2} - d_5^{(1)}\right) Z_o(\zeta \underline{\alpha}^{(1)} \bar{a}),$$

$$C_{12} = 2\, d_6^{(1)}\, \zeta^2\, \underline{\alpha}^{(1)}\, \bar{a}\, W_1 (\zeta \underline{\alpha}^{(1)} \bar{a}) -$$

[35]

$$- \frac{\zeta^3 d_4^{(1)} \bar{a}^2}{d_5^{(1)} + d_4^{(1)}} \left(d_1^{(1)} \varepsilon_1^{(1)}\, \underline{\alpha}^{(1)2} + d_5^{(1)} \frac{\Omega^2}{a^{(1)2} \zeta^2} - d_5^{(1)}\right) W_o (\zeta \underline{\alpha}^{(1)} \bar{a}),$$

$$C_{13} = 2\, d_6^{(1)} (d_5^{(1)} + d_4^{(1)})\, \varepsilon_2^{(1)}\, \zeta \underline{\beta}^{(1)} \bar{a}\;\; Z_1 (\zeta \underline{\beta}^{(1)} \bar{a}) -$$

$$- d_4^{(1)} \zeta^2 \bar{a}^2 \left(d_1^{(1)} \varepsilon_1^{(1)}\, \underline{\beta}^{(1)2} + d_5^{(1)} \frac{\Omega^2}{a^{(1)2} \zeta^2} - d_5^{(1)}\right) Z_o(\zeta \underline{\beta}^{(1)} \bar{a}),$$

$$C_{14} = 2\, d_6^{(1)}(d_5^{(1)} + d_4^{(1)})\zeta\underline{\beta}^{(1)}\bar{a}\, W_1\,(\zeta\underline{\beta}^{(1)}\bar{a}) -$$

$$- d_4^{(1)}\zeta^2\,\bar{a}^2\,(d_1^{(1)}\varepsilon_1^{(1)}\underline{\beta}^{(1)2} + \frac{d_5^{(1)}\,\Omega^2}{a^{(1)2}\,\zeta^2} - d_5^{(1)})\, W_o\,(\zeta\underline{\beta}^{(1)}\bar{a}),$$

$$C_{21} = - \frac{d_4^{(1)}\zeta^2\bar{a}}{d_5^{(1)} + d_4^{(1)}}\,(d_1^{(1)}\varepsilon_1^{(1)}\underline{\alpha}^{(1)2} - \frac{d_4^{(1)}\,\Omega^2}{a^{(1)2}\,\zeta^2} - d_5^{(1)})\varepsilon_1^{(1)}\zeta\underline{\alpha}^{(1)}\bar{a}\, Z_1(\zeta\underline{\alpha}^{(1)}\bar{a}),$$

$$C_{22} = - \frac{d_4^{(1)}\zeta^2\bar{a}}{d_5^{(1)} + d_4^{(1)}}\,(d_1^{(1)}\varepsilon_1^{(1)}\underline{\alpha}^{(1)2} - \frac{d_4^{(1)}\,\Omega^2}{a^{(1)2}\,\zeta^2} - d_5^{(1)})\zeta\underline{\alpha}^{(1)}\bar{a}\, W_1(\zeta\underline{\alpha}^{(1)}\bar{a}),$$

$$C_{23} = - d_4^{(1)}\zeta\bar{a}(d_1^{(1)}\varepsilon_2^{(1)}\underline{\beta}^{(1)2} - \frac{d_4^{(1)}\,\Omega^2}{a^{(1)2}\,\zeta^2} - d_5^{(1)})\varepsilon_2^{(1)}\zeta\underline{\beta}^{(1)}\bar{a}\, Z_1(\zeta\underline{\beta}^{(1)}\bar{a}),$$

$$C_{24} = - d_4^{(1)}\zeta\bar{a}(d_1^{(1)}\varepsilon_2^{(1)}\underline{\beta}^{(1)2} - \frac{d_4^{(1)}\,\Omega^2}{a^{(1)2}\,\zeta^2} - d_5^{(1)})\zeta\underline{\beta}^{(1)}\bar{a}\, W_1\,(\zeta\underline{\beta}^{(1)}\bar{a}),$$

$C_{3,i}$, $C_{4,i}$ $(i = 1,2,3,4) = C_{2,i}$, $C_{1,i}$ $(i = 1,2,3,4)$ with $\bar{a}$ replaced by $\bar{b}$,

$C_{3,i}$, $C_{4,i}$ $(i = 5,6,7,8) = C_{2,i}$, $C_{1,i}$ $(i = 1,2,4,5)$ with $\bar{a}$ replaced by $\bar{b}$

and (1) by (2),

$$C_{51} = - \varepsilon_1^{(1)}\zeta^2\,\underline{\alpha}^{(1)}\bar{b}\, Z_1\,(\zeta\underline{\alpha}^{(1)}\bar{b})\,, \qquad [35\ \text{cont}]$$

$$C_{52} = - \zeta^2\,\underline{\alpha}^{(1)}\bar{b}\, W_1\,(\zeta\underline{\alpha}^{(1)}\bar{b}),$$

$$C_{53} = - (d_5^{(1)} + d_4^{(1)})\,\varepsilon_2^{(1)}\zeta\underline{\beta}^{(1)} Z_1(\zeta\underline{\beta}^{(1)}\bar{b}),$$

$$C_{54} = -\,(d_5^{(1)} + d_4^{(1)})\ \zeta\underline{\beta}^{(1)}\bar{b}\ W_1\ (\zeta\underline{\beta}^{(1)}\bar{b}),$$

$$C_{61} = \frac{\zeta^2\,\bar{b}}{d_5^{(1)} + d_4^{(1)}}\ (d_1^{(1)}\,\varepsilon_1^{(1)}\ \underline{\alpha}^{(1)^2} - \frac{d_4^{(1)}\,\Omega^2}{a^{(1)^2}\,\zeta^2} + d_4^{(1)})\ Z_o\ (\zeta\underline{\alpha}^{(1)}\bar{b}),$$

$$C_{62} = \frac{\zeta^2\,\bar{b}}{d_5^{(1)} + d_4^{(1)}}\ (d_1^{(1)}\varepsilon_1^{(1)}\ \underline{\alpha}^{(1)^2} - \frac{d_4^{(1)}\,\Omega^2}{a^{(1)^2}\,\zeta^2} + d_4^{(1)})\ W_o\ (\zeta\underline{\alpha}^{(1)}\bar{b}),$$

$$C_{63} = \zeta\bar{b}\ (d_1^{(1)}\varepsilon_2^{(1)}\ \underline{\beta}^{(1)^2} - \frac{d_4^{(1)}\,\Omega^2}{a^{(1)^2}\,\zeta^2} + d_4^{(1)})\ Z_o\ (\zeta\underline{\beta}^{(1)}\bar{b}),$$

[35 cont]

$$C_{64} = \zeta\bar{b}\ (d_1^{(1)}\,\varepsilon_2^{(1)}\ \underline{\beta}^{(1)^2} - \frac{d_4^{(1)}\,\Omega^2}{a^{(1)^2}\,\zeta^2} + d_4^{(1)})\ W_o\ (\zeta\underline{\beta}^{(1)}\bar{b}),$$

$C_{5,j}$, $C_{6,j}$ $(j = 5,6,7,8) = C_{5,i}$, $C_{6,i}$ $(i = 1,2,3,4)$ with (1) replaced by (2),

$C_{7,j}$, $C_{8,j}$ $(j = 5,6,7,8) = C_{6,i}$, $C_{5,i}$ $(i = 1,2,3,4)$ with $\bar{b}$ replaced by $\bar{c}$ and (1) replaced by (2),

$C_{7,j}$, $C_{8,j}$ $(j = 9,10,11,12) = C_{6,i}$, $C_{5,i}$ $(i = 1,2,3,4)$ with $\bar{b}$ replaced by $\bar{c}$ and (1) replaced by (3).

$C_{9,j}$, $C_{10,j}$ $(j = 5,6,7,8) = C_{1,j}$, $C_{2,j}$ $(j = 1,2,3,4)$ with $\bar{a}$ replaced by $\bar{c}$ and (1) by (2).

$C_{9,j}$, $C_{10,j}$ $(j = 9,10,11,12) = C_{1,j}$, $C_{2,j}$ $(j = 1,2,3,4)$ with $\bar{a}$ replaced by $\bar{c}$ and (1) by (3).

$C_{11,j}$, $C_{12,j}$ (j = 9,10,11,12) = $C_{2,j}$ $C_{1,j}$ (j = 1,2,2,4) with $\bar{a}$ [35 cont]

replaced by $\bar{d}$ and (1) by (3).

For given material properties and shell geometry, the frequency equation [34] is a transcendental relation between the non-dimensionalized frequency Ω and the wave number ζ. For any value of ζ, the frequency equation will yield an infinite number of values of Ω, each corresponding to a different mode of wave propagation.

The frequency equation may be specialized to give the correct formulation for solid rods ($\bar{a}$ = 0). In this case, the boundary conditions at the inner surface (See Eq.[33])are omitted. Furthermore, in order for the displacement field in layer (1) to remain finite at $\bar{r}$ = 0, it must contain no Bessel functions of the second kind. This implies that the constants of integration $B_j^{(1)}$(j = 1,2) must vanish. Consequently, the frequency equation for axisymmetric waves in three-layered, transversely isotropic rods may be obtained by deleting the first and second rows and the second and fourth columns of equation [34].

For "more anisotropic" materials, within the range of values of Ω and ζ for which the values of $\underline{p}$ and $\underline{q}$ are complex, the frequency equation will contain complex elements. Inasmuch as $\underline{q}$ is the negative complex conjugate of $\underline{p}$, the following relations are valid

$$J_n(\underline{q}\,\bar{r}) = (-1)^n \quad J_n^*(\underline{p}\,\bar{r}),$$
$$Y_n(\underline{q}\,\bar{r}) = (-1)^n \quad Y_n^*(\underline{p}\,\bar{r}),$$ [36]

where J_n^* and Y_n^* are the complex conjugates of J_n and Y_n.

If the i^{th} layer of the cylinder is "more anisotropic," and if m is sufficiently small so that $\underline{\alpha}$ and $\underline{\beta}$ are given by equation [19], then the four columns of Eq. [34] relating to the i^{th} layer are complex. It can be shown, that with the exception of a constant multiplication factor, columns (4i-n) and (4i-n+2) (n=2,3) are, element by element, complex conjugate pairs. For instance, if layer 1 is complex (i=1), the frequency equation may be written as

$$|C_{jk}| = |G_{j1}\ G_{j2}\ G_{j3}\ G_{j4}\ C\ | = 0\ , \qquad [37]$$

where C represents the last eight columns of the determinant and G_{jk} (k=1,2,3,4) are the complex first four columns of C_{jk}. These may be written as

$$G_{j1} = \tilde{C}_{j1} + i\,\tilde{C}_{j3}, \qquad G_{j3} = \tilde{C}_{j1} - i\,\tilde{C}_{j3}\ ,$$
$$(j=1,2,\ \ldots,12) \qquad [38]$$
$$G_{j2} = \tilde{C}_{j2} + i\,\tilde{C}_{j4}, \qquad G_{j4} = \tilde{C}_{j2} - i\,\tilde{C}_{j4}.$$

Substitution into equation [37] and expansion yields

$$|C_{jk}| = -4\ |\ \tilde{C}_{j1}\ \tilde{C}_{j2}\ \tilde{C}_{j3}\ \tilde{C}_{j4}\ C\ | = 0\ . \qquad [39]$$

Thus, the problem of evaluating the frequency equation for waves traveling in shells made of "more anisotropic" materials is essentially identical to that of evaluating the frequency equations for waves traveling in shells made of "less anisotropic" materials.

WAVES WITH INFINITELY LONG AXIAL WAVE LENGTHS

When the axial wave number vanishes, the displacement field of the cylinder is independent of the axial coordinate. The radial wave numbers reduce to

$$p_c^{(i)^2} = \frac{\Omega^2}{a^{(i)^2}} \qquad \text{and} \qquad q_c^{(i)^2} = \frac{d_4^{(i)}}{d_1^{(i)}} \, p_c^{(i)^2} , \qquad [40]$$

and the frequency determinant can be written as the product of the following two determinants

$$\underset{\zeta \to 0}{\text{Limit}} \; |C_{ij}| = D_1 \cdot D_2 = 0, \qquad [41]$$

where

$$D_1 = \begin{vmatrix} C_{21} & C_{22} & 0 & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{35} & C_{36} & 0 & 0 \\ C_{61} & C_{62} & C_{65} & C_{66} & 0 & 0 \\ 0 & 0 & C_{75} & C_{76} & C_{79} & C_{7,10} \\ 0 & 0 & C_{10,5} & C_{10,6} & C_{10,9} & C_{10,10} \\ 0 & 0 & 0 & 0 & C_{11,9} & C_{11,10} \end{vmatrix} , \qquad [42]$$

and

$$D_2 = \begin{vmatrix} C_{13} & C_{14} & 0 & 0 & 0 & 0 \\ C_{43} & C_{44} & C_{47} & C_{48} & 0 & 0 \\ C_{53} & C_{54} & C_{57} & C_{58} & 0 & 0 \\ 0 & 0 & C_{87} & C_{88} & C_{8,11} & C_{8,12} \\ 0 & 0 & C_{97} & C_{98} & C_{9,11} & C_{9,12} \\ 0 & 0 & 0 & 0 & C_{12,11} & C_{12,12} \end{vmatrix} . \qquad [43]$$

The equation $D_1 = 0$ yields the cut-off frequencies of axisymmetric longitudinal shear vibrations involving only axial displacement. In this case, the motion is uncoupled equivoluminal, and the displacement components are

$$\bar{u}_r^{(i)} = 0,$$
$$\bar{u}_z^{(i)} = -p_c^{(i)^2} \left[A_1^{(i)} J_0 (p_c^{(i)}\bar{r}) + B_1^{(i)} Y_0 (p_c^{(i)}\bar{r}\right] \sin \Omega \bar{t}. \qquad [44]$$

Notice that, as in the case of isotropic shells, the motion depends only on the elastic constants $d_4^{(i)} (i = 1, 2, 3)$.

The equation $D_2 = 0$ represents plane strain extensional motion involving only radial displacements

$$\bar{u}_r^{(i)} = q_c^{(i)} [A_2^{(i)} J_1 (q_c^{(i)}\bar{r}) + B_2^{(i)} Y_1 (q_c^{(i)}\bar{r})] \cos \Omega \bar{t}. \qquad [45]$$
$$\bar{u}_z^{(i)} = 0.$$

This motion is independent of the elastic constants $d_3^{(i)}$ and $d_5^{(i)}$ and consequently is independent of the axial Young's moduli of the three materials.

NUMERICAL ANALYSIS

A computer program has been written for numerical evaluation of the frequency equations. The program first computes the cut-off frequencies on the basis of Eqs. [42] and [43], and utilizes them as starting values to trace the branch curves of Eq. [34] on the Ω - ζ plane. For each assumed value of ζ, the frequency, Ω, is incremented by a specified amount $\Delta\Omega$ until a change in the sign of the determinant occurs. This indicates a root between the last two values of Ω. An interval halfing procedure is then executed which brackets the root to the required (pre-set) accuracy. Subsequently, the value of ζ is incremented by a pre-assigned increment $\Delta\zeta$ and, starting with a new value of Ω, (computed from the slope of the two previously established points on the branch) the process is repeated and new roots are established until each branch of the frequency equation has been traced up to a pre-assigned value of ζ. For each tested value of Ω and ζ, the program uses the appropriate form of the frequency equation depending on whether the radial wave numbers p and q are real, imaginary or complex.

The effect of the ratio of elastic constants $c_{44}^{(2)}$ and $c_{11}^{(2)}$ (for constant a ratio of $c_{11}^{(2)}/\ c_{12}^{(2)} = 7/3$) on the cut-off frequencies of the first few modes is illustrated in Figs. 3 through 6. In these figures, the outside and the inside layer were chosen as Boron/Epoxy (see Table 1). This material is "less anisotropic". The results shown are valid for any values of $c_{33}^{(2)}$ and $c_{13}^{(2)}$ inasmuch as the cut-off frequencies[equations [42] and [43]]are independent of these elastic constants. As expected, the

frequencies of the longitudinal shear modes are independent of the elastic constants $c_{11}^{(2)}$ and c_{12}^{2}. Moreover, as can be seen from Fig. 3 and 4, the frequency of the first longitudinal shear mode is not noticeably effected by changes in the axial shear modulus $c_{44}^{(2)}$. The frequency for this mode is slightly larger for rods (Fig. 4) than for thick-walled shells, i.e. $H/R = 1.0$ (Fig. 3). For values of $H/R < 1.0$ the effect of H/R on the frequency of this mode is negligible. Thus, for thin sandwich shells the frequency of the first longitudinal shear mode is only effected by changes in the density ratio of the layers. This result is interesting inasmuch as this frequency is employed in establishing the correction factor in Timoshenko-type shell theories.

TABLE 1. - MATERIAL CONSTANTS

Material	Pounds Per Square Inch $(10)^6$					$\rho(\mathrm{lb/in}^3)$
	c_{11}	c_{12}	c_{33}	c_{44}	c_{13}	
Aluminum	13.46	5.76	13.46	3.85	5.76	0.100
Boron/Epoxy Composite	3.28	1.19	30.4	1.00	0.995	0.075
Beryllium	4.24	0.388	4.88	2.36	0.203	0.067

Figure 7 shows the frequency spectrum of a sandwich shell made of an aluminum core and fiber-reinforced composite facings made of an epoxy matrix reinforced by unidirectional boron fibers. Figure 8 shows the frequency spectrum of a sandwich shell made of an aluminum core and beryllium plate facings. The mechanical properties of the layers of these shells are given in Table 1. The aluminum is isotropic, whereas the Boron/Epoxy composite and the beryllium are less "anisotropic" and "more anisotropic" respectively. The Ω-ζ plane may be sub-divided into sectors by lines $\Omega = b^{(i)}\zeta$, $\Omega = a^{(i)}\zeta$, and $\Omega = \frac{m_j^{(i)}}{\sqrt{c_{44}}}\zeta$ $(i = 1, 2, 3;\ j = 1, 2)$. Throughout each sector the radial wave numbers p and q retain their real, imaginary or complex character and, consequently, the form of the frequency equation, in each sector, does not change. It can be seen that the behavior of the frequency lines of the lowest two modes of wave propagation in the shell having fiber-reinforced composite facings differs considerably from those for the shell with the beryllium facings; those of the latter cross into the region of complex p and q. The frequency lines of the higher modes for the two shells are comparable for large wave numbers; they appear to become parallel to the $\Omega = a^{(2)}\zeta$ line. For smaller wave numbers, within a certain range of ζ, the frequency lines for the shell with fiber-reinforced composite facings become nearly parallel to the $\Omega = b^{(b)}\zeta$ line. This tendency is not apparent in the spectrum for the shell with the beryllium facings.

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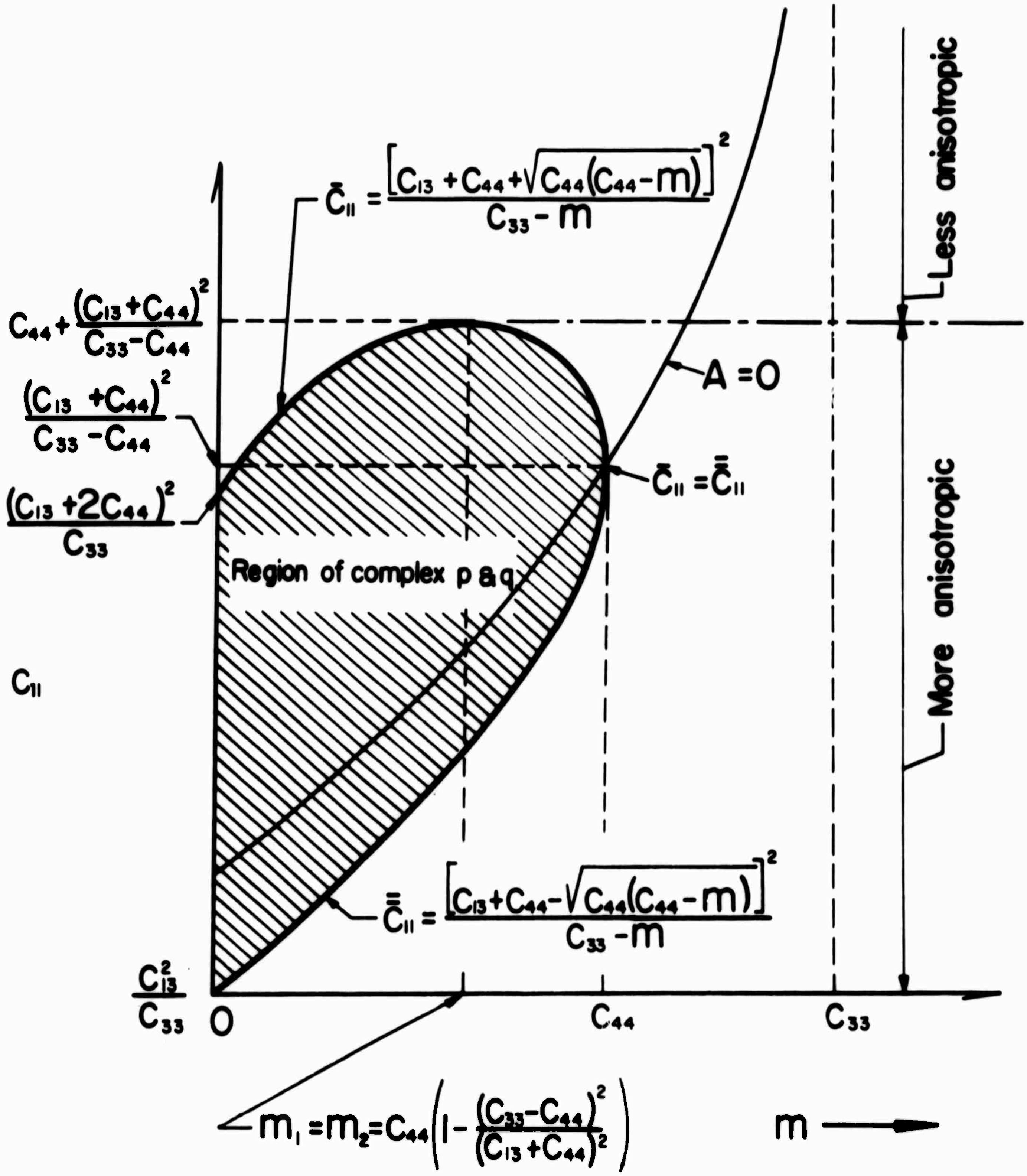

Fig. 1 Region of complex radial wave numbers.

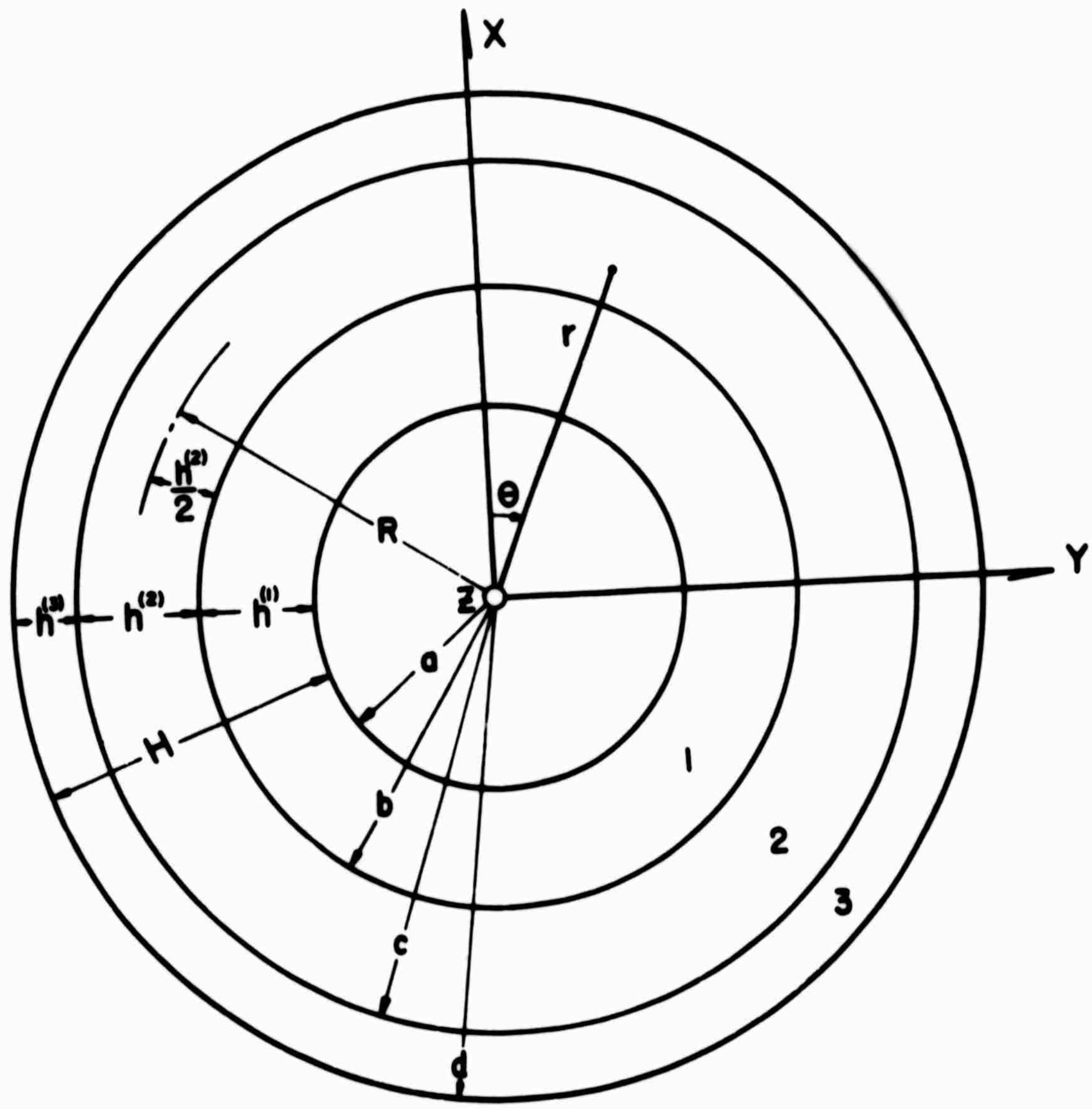

Fig. 2 Cross-section of the three-layer cylinder

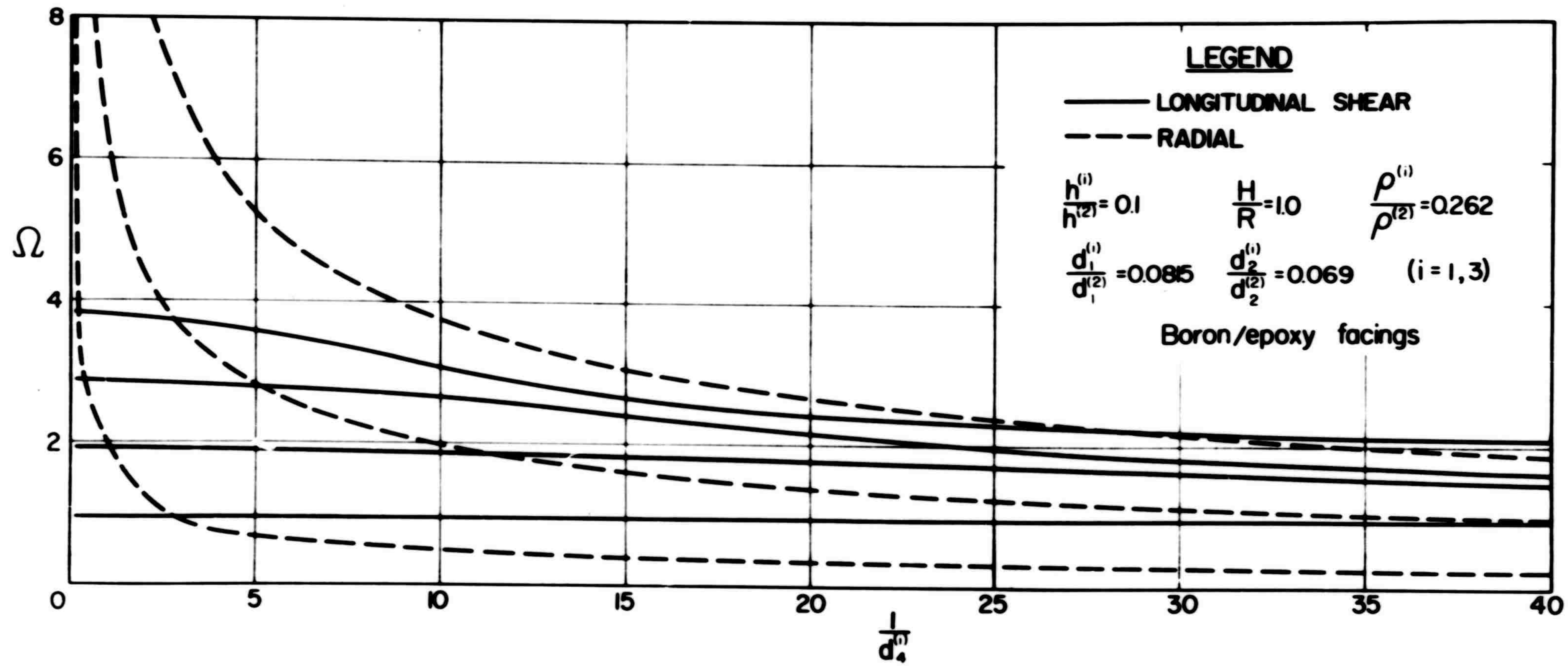

Fig. 3 Variation of cut-off frequencies as function of $\frac{1}{d_4^{(i)}}$.

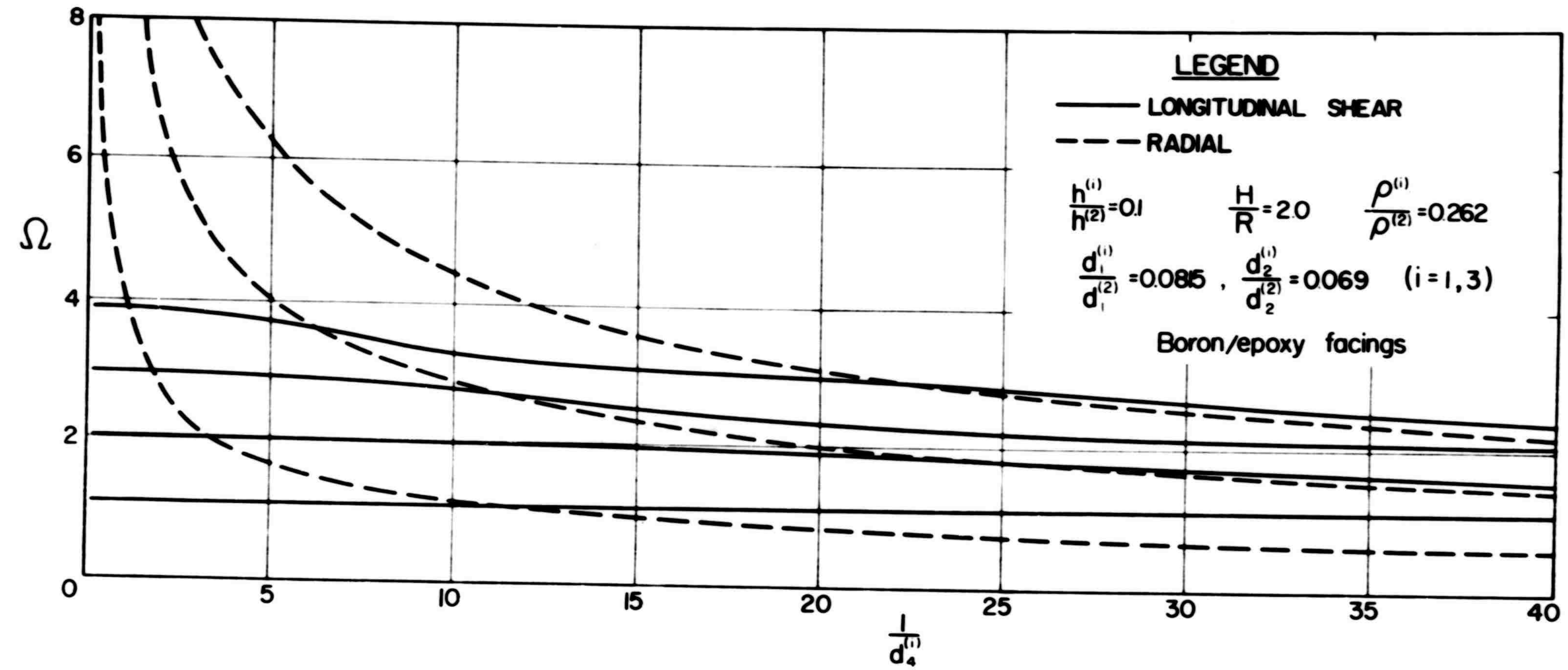

Fig. 4 Variation of cut-off frequencies as functions of $\frac{1}{d_4^{(i)}}$

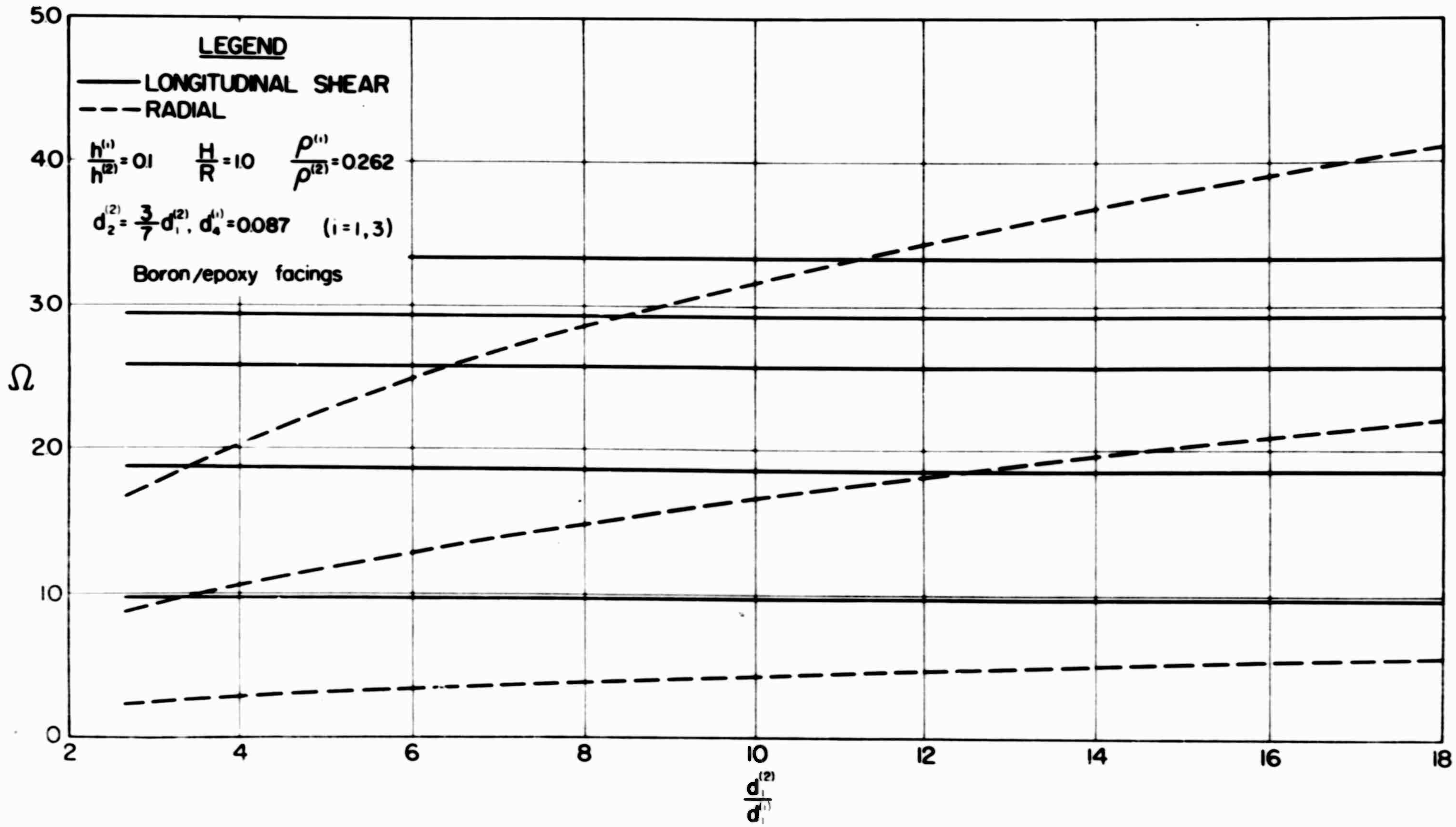

Fig. 5 Variation of cut-off frequencies as functions of $\frac{d_1^{(2)}}{d_1^{(i)}}$

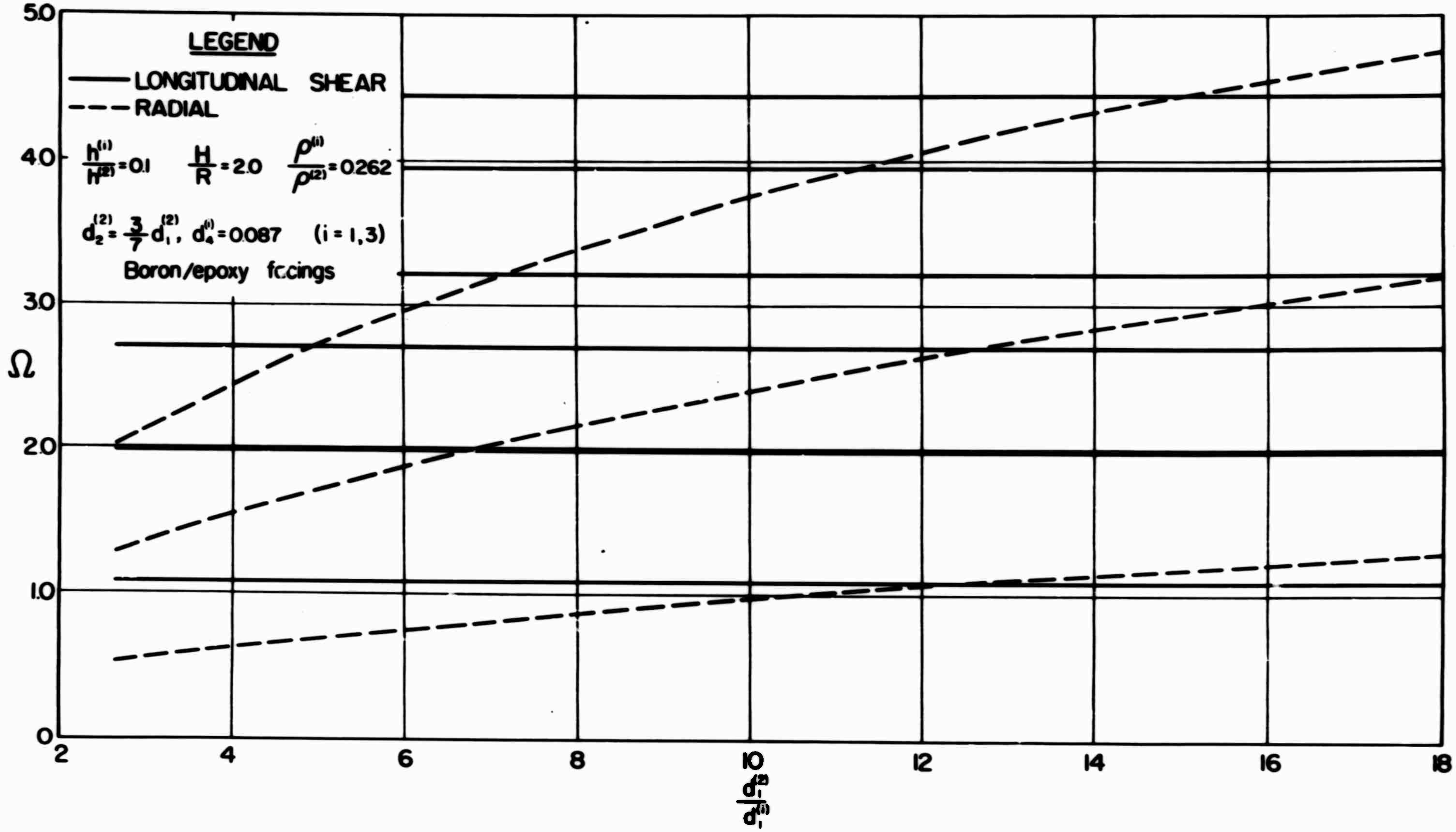

Fig. 6 Variation of cut-off frequencies as functions of $\frac{d_1^{(2)}}{d_1^{(i)}}$

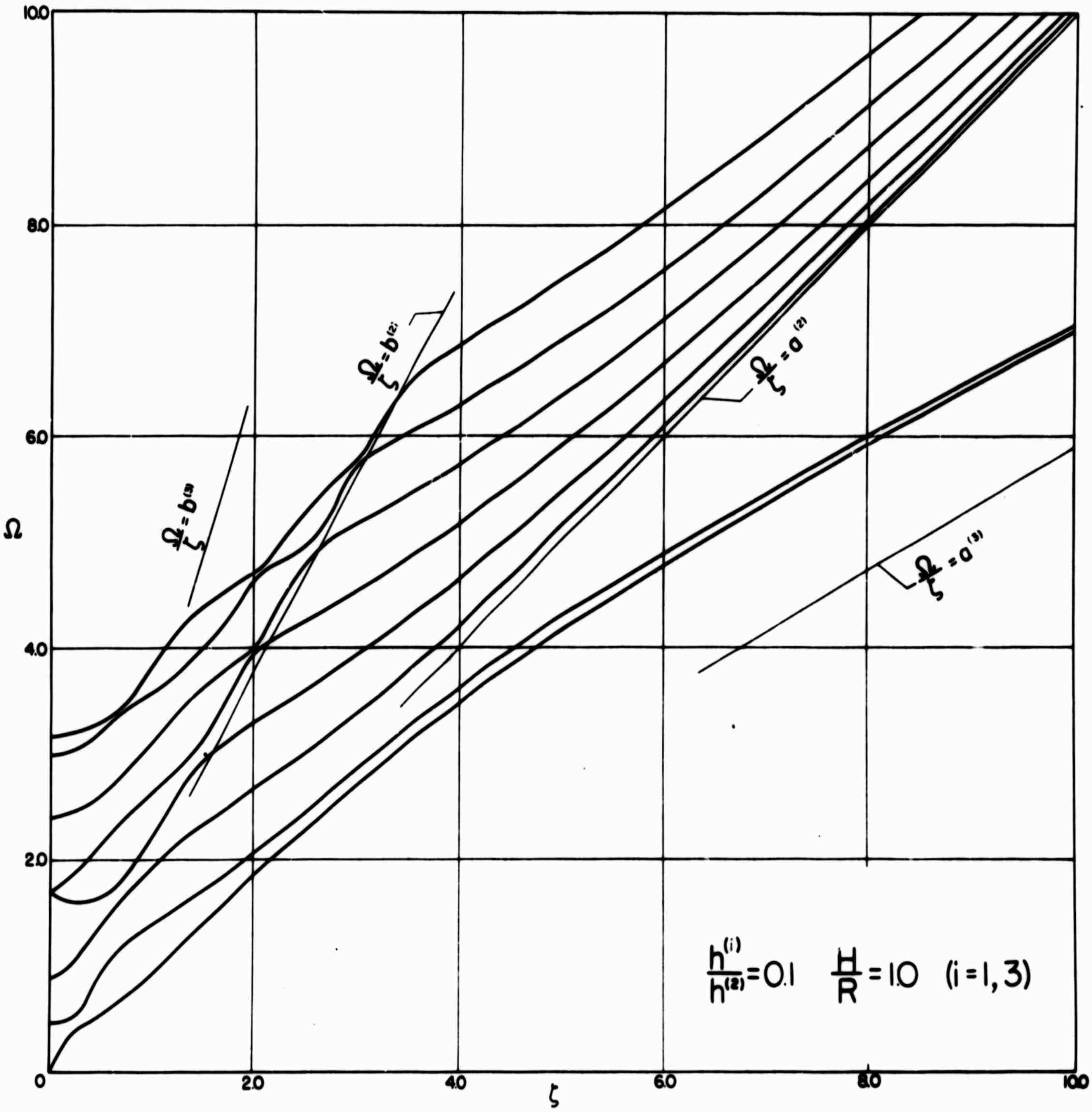

Fig. 7 Frequency spectrum for a shell with aluminum core and fiber-reinforced composite facings.

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13. ABSTRACT

The propagation of infinite trains of symmetric harmonic waves traveling in infinitely long, right circular cylindrical shells is investigated on the basis of the three-dimensional theory of elasticity. The shells are assumed made of three concentric, transversely isotropic cylinders each of different materials bonded perfectly at their interfaces. The frequency equation is established by representing the displacement field in each cylinder in terms of potential functions and satisfying the Navier equations of motion and the boundary and interface conditions of the cylinder.

The frequency equation has been programmed for numerical evaluation on an IBM 7044/7094 DCS computer, and the influence of the mechanical properties of the layers on the frequencies of the first few modes is investigated.

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